Comment on "Torque on a Satellite Due to Gravity Gradient and Centrifugal Force"

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IN Ref. 1, it is stated that on an orbiting satellite there exists, in addition to the gravity-gradient torque, an "external" torque caused by the gradient in centrifugal force. Considering this latter torque as an external torque leads to some question as to its use in the equations of motion. Considering only the motion relative to the satellite center of mass, the equations of motion in matrix form are

$$\{T\} = (d/dt)[I]\{\omega\} \tag{1}$$

where $\{T\}$ is the vector representing the net external torque, $[I]$ is the inertia tensor taken relative to the center of mass, and $\{\omega\}$ is the vector representing the angular velocity of the satellite body relative to inertial space, and the time derivative indicated is taken relative to inertial space.

Since $\{T\}$ represents the external torque, it would seem that, in addition to the gravity-gradient torque, the centrifugal gradient torque mentioned in Ref. 1 also should be incorporated into $\{T\}$. However, this is erroneous, since the centrifugal gradient torque is accounted for by the right-hand side of Eq. (1). If $[I]$ and $\{\omega\}$ are written in terms of a body-fixed, principal-axis system and if the vector operator relating the inertial time derivative d/dt to the time derivative in a body-fixed system $\delta/\delta t$ is applied to the right-hand side of (1), there results

$$\{T\} = [I](\delta\{\omega\}/\delta t) + \{\omega\} \times ([I]\{\omega\}) \tag{2}$$

where $\{A\} \times \{B\}$ represents the cross product of the vectors A and B. It appears that it is the term $\{\omega\} \times ([I]\{\omega\})$ that Ref. 1 considers to be an external torque. In reality, it does not represent an external torque, but is simply a term, which is due to the rotation of the body-fixed coordinate system relative to inertial space. If the problem were analyzed using Eq. (1) directly, these terms would not appear explicitly.

The term $\{\omega\} \times ([I]\{\omega\})$ can be shown to be equal to the right-hand side of Eq. (2) in Ref. 1 by choosing the satellite geometry and axis system of Ref. 1. A symmetric satellite in circular orbit is considered with a body-fixed, principal-axis system as shown in Fig. 1.

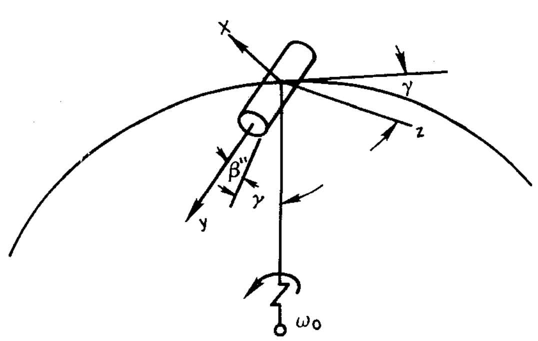

Fig. 1 Definition of coordinate system (the z axis lies in the orbital plane, β'' is the angle between the y axis and the orbital plane, and γ is the angle between the z axis and the tangent to the orbit).

The angle β'' is identical to the one used in Ref. 1. Further, if the body motion is restricted so that the angles β'' and γ are fixed, and the z axis always lies in the orbital plane, then the angular velocity of the body-fixed axis system relative to inertial space lies in the x-y plane and is given as

$$\{\omega\} = \begin{Bmatrix} \omega_0 \cos\beta'' \\ \omega_0 \sin\beta'' \\ 0 \end{Bmatrix}$$

where ω_0 is the orbital frequency. Further $I_z = I_x = I_t$, and $I_y = I_s$. With these restrictions, the term $\{\omega\} \times ([\mathrm{I}]\{\omega\})$ becomes

$$(w_0{}^2/2) \sin 2\beta''(I_s - I_t)$$

which is identical to the right-hand side of Eq. (2) in Ref. 1. In fact, taking the analysis one step further, Eq. (2) indicates that, to maintain the restricted motion considered, the net external torque acting on the body (comprising gravity-gradient torques, corrective torques from an attitude-control system, aerodynamic torques, etc.) must be

$$\{T\} = \begin{Bmatrix} 0 \\ 0 \\ (\omega_0{}^2/2) \sin 2\beta''(I_s - I_t) \end{Bmatrix}$$

In conclusion, it appears that the results presented in Ref. 1 do not, in reality, represent external torques and, further, are restricted to cases where a vehicle axis is continually oriented toward the earth center, thereby requiring a body rate equal to the orbit rate ω_0, normal to the orbit plane. It is this body rate, not the orbit motion, which yields what appears to be an "external" torque in an equation of the form $T_x = I_x(d\omega_x/dt)$. However, by using the complete dynamical equations for principal-body axes [e.g., $T_x = I_x(d\omega_x/dt) + \omega_y\omega_z(I_z, - I_y)$, etc.], the alleged "centrifugal-force gradient" torques are seen to arise naturally from the complete equations of motion and, therefore, should not be considered as separate external torques.

Reference

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An Improved Error Estimate for Sychev's Entropy Layer Solution

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THE entropy layer adjoining a blunted slender body in hypersonic flow causes failure of the equivalence principle similarity solution,[1] even far downstream. Sychev[2] has modified the equivalence solution, providing a theory with errors at most of order τ^2 where τ is the local shock slope. By comparing Sychev's simple theory with more complicated ones, we find most field quantities to agree to order τ^2.

We consider the inverse problem, with a power-law shock wave $y_s = cx^n$, $n \leqslant 1$. Sychev transformed the governing equations to von Mises variables (x, ψ) and found approximate integration possible, neglecting terms of order τ^2.

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Table 1 Various authors' notation[a]

Meaning	Sychev	Yakura	Lee
Shock exponent	n	n	n
Stream function	ψ	ψ	$\psi(4n^2)^{(1-n)/n}$
ψ/x^{2n}	$\eta(\lambda)$	$\omega(\lambda)$	$\omega(z)$
y/y_s[b]	λ	λ	z
v/v_s	$f(\lambda)$	$f(\lambda)$	$\phi(z)\,(\gamma+1)/2$
ρ/ρ_s	$g(\lambda)$	$g(\lambda)$	$\beta(z)\left(\dfrac{\gamma-1}{\gamma+1}\right)$
p/p_s	$h(\lambda)$	$h(\lambda)$	$F(z)\,(\gamma+1)/2$

[a] Quantities on each line are identical.
[b] Subscript s means evaluated at the shock.

Table 2 Various authors' results, expressed in common symbols

Quantity	Sychev[a]
$\eta(\lambda)$	$\exp -2(\gamma+1)\int_1^{\lambda} [2f(\lambda)-(\gamma+1)\lambda]^{-1}\,d\lambda$
$p(x, \psi)$	$[4n^2/(\gamma+1)]h(\lambda)x^{2(n-1)}$
$\rho(x, \psi)$	$[(\gamma+1)/(\gamma-1)]G^{-1}(x, \psi)$ [e]
$u(x, \psi)$	$\{1-[8n^2\gamma/(\gamma+1)^2]h(\lambda)G(x, \psi)x^{2(n-1)}\}^{1/2}$ [e]
$v(x, \psi)$	$[2^{3/2}/(\gamma+1)]nf(\lambda)x^{n-1}$
$y(x, \psi)$	$2^{1/2}x^n\left[1+\dfrac{\gamma-1}{\gamma+1}\int_1^{\eta}\dfrac{G(x, \eta)}{u(x, \eta)}\,d\eta\right]^{1/2}$
where	$G(x, \eta) \equiv [h(\lambda)(\eta^{(1-n)/n}+2n^2x^{2(n-1)})]^{-1/\gamma}$

Quantity	Yakura
$\eta(\lambda)$	$[\lambda g(\lambda)/(\gamma-1)][(\gamma+1)\lambda-2f(\lambda)]$ [b]
$p(x, \psi)$	$\left[\dfrac{4n^2}{(\gamma+1)}\right]h(\lambda)x^{2(n-1)}+\ldots$ [b]
$\rho(x, \psi)$	$\left[\dfrac{(\gamma+1)}{(\gamma-1)}\right]\{g(\lambda)+h(0)^{1/\gamma}[(2n^2+\psi^{(1-n)/n})^{1/\gamma}-(\psi^{(1-n)/n})^{1/\gamma}]x^{(2/\gamma)(n-1)}\}+\ldots$ [b]
$u(x, \psi)$	$1-\dfrac{4n^2\gamma}{(\gamma+1)^2}\left\{[(2n^2+\psi^{(1-n)/n})^{-1/\gamma}-(\psi^{(1-n)/n})^{-1/\gamma}]\left(\dfrac{h(0)}{x^{2(1-n)}}\right)^{(\gamma-1)/\gamma}+\left[\dfrac{f^2(\lambda)}{\gamma}+\dfrac{h(\lambda)}{g(\lambda)}\right]x^{2(n-1)}\right\}+\ldots$ [b]
$v(x, \psi)$	$\dfrac{2^{1/2}}{\gamma+1}f(\lambda)x^{-1/2}+\dfrac{1}{2[\gamma(\gamma+1)]^{1/2}}\left(\dfrac{2}{h(0)}\right)^{1/2\gamma}\times[(2\psi+1)^{(\gamma-1)/2\gamma}-(2\psi)^{(\gamma-1)/2\gamma}]x^{(1-2\gamma)/2\gamma}+\ldots$
$y(x, \psi)$	$2^{1/2}\lambda x^n+[\gamma/(\gamma+1)]^{1/2}\{2/[h(0)]\}^{1/2\gamma}\times[(2\psi+1)^{(\gamma-1)/2\gamma}-(2\psi)^{(\gamma-1)/2\gamma}]x^{1/2\gamma}+\ldots$ [c]

Quantity	Lee
$\eta(\lambda)$	Same results as Yakura
$p(x, \psi)$	Same results as Yakura
$\rho(x, \psi)$	Same results as Yakura
$u(x, \psi)$	Same results as Yakura
$v(x, \psi)$	$\dfrac{2^{3/2}}{\gamma+1}nf(\lambda)x^{n-1}+\dfrac{\gamma-1}{\gamma+1}\,\dfrac{2(1-n)-\gamma n}{2^{1/2}\gamma\lambda_b}\times(1/h(0))^{1/\gamma}I(\psi; \gamma, n)\,x^{-[1+n+(2n/\gamma)-(2/\gamma)]}+\ldots$ [d]
$y(x, \psi)$	$2^{1/2}\lambda x^n+[(\gamma-1)/(\gamma+1)][1/2^{1/2}\lambda_b](1/h(0))^{1/\gamma}\times I(\psi; \gamma, n)\,x^{-n+(2/\gamma)(1-n)}+\ldots$ [d]
where	$I(\psi; \gamma, n) \equiv \int_{\psi}^{\infty}[(\psi^{(1-n)/n})^{-1/\gamma}-(2n^2+\psi^{(1-n)/n})^{-1/\gamma}]\,d\psi$

[a] Specialized to $\nu = 1$, $c = 2^{1/2}$, $U_\infty = 1$, $\rho_\infty = 1$ for comparison.
[b] For $n = \frac{1}{2}$, $(\gamma+2)/(3\gamma+2) < n \leqslant (\gamma+1)/(2\gamma+1)$.
[c] For $n = \frac{1}{2}$.
[d] For $(\gamma+2)/(3\gamma+2) < n \leqslant (\gamma+1)/(2\gamma+1)$, subscript b means evaluated at body ($\psi = 0$).
[e] $G(x, \psi)$ should be read $G(x, \eta) \equiv G(x, \psi/x^{2n})$, $G(x, \eta)$ defined within the table.

Yakura[3] solved the axisymmetric case for $n = \frac{1}{2}$ using the method of matched asymptotic expansions. Lee[4] extended Yakura's solution to cover the range $[(\gamma+2)/(3\gamma+2)] < n \leqslant [(\gamma+1)/(2\gamma+1)]$. Below this range Lee found the matched expansions slowly convergent; above it Sychev showed corrections for the thickness of the entropy layer unnecessary. We now expand Sychev's results far downstream ($x \to \infty$) and compare them to Yakura's and Lee's composite expansions.

Yakura and Lee considered the case of axisymmetric flow (Sychev's $\nu = 1$) and set $\rho_\infty = U_\infty = 1$. They also set $c = 2^{1/2}$ in order to compare their results with Sychev's numerical results. Sychev and Yakura used the notation of Sedov[5]; Lee used that of Mirels.[6] Table 1 gives the identities used in reducing all the results to a single notation. Table 2 displays these reduced results. We proceed by taking the limit of $x \to \infty$ in Sychev's results, keeping ψ or $\eta \equiv \psi/x^{2n}$ constant. These give, respectively, "inner" and "outer" expansions. Our composite expansions are formed by adding these and subtracting out their overlap. In doing this, careful use must be made of the binomial theorem to get uniformly valid roots and powers. Also we must use the relations

$$h(\lambda)\eta^{(1-n)/n}(\lambda) = g^{\gamma}(\lambda) \qquad \text{(for nonvanishing } \lambda)$$

$$\lim_{\substack{x\to\infty\\ \eta \text{ fixed}}} \int_1^{\eta} [h(\lambda)(2n^2x^{2(n-1)}+\eta^{(1-n)/n})]^{-1/\gamma}d\eta = [(\gamma+1)/(\gamma-1)]\,(\lambda^2-1)$$

$$\lim_{\substack{x\to\infty\\ \psi \text{ fixed}}} \int_1^{\eta} [h(\lambda)(2n^2x^{2(n-1)}+\eta^{(1-n)/n})]^{-1/\gamma}d\eta = -h(0)^{-1/\gamma}\,x^{(2/\gamma)(1-n)-2n}\int_{\psi}^{\infty}[2n^2+\psi^{(1-n)/n}]^{-1/\gamma}\,d\psi$$

The resulting expansions for the pressure p, density ρ, and coordinate y are identical to those displayed in Table 2 for Yakura† and Lee. The expansion for the freestream component of velocity u is in agreement except that it has $[h(\lambda)/g(\lambda)]$ where Yakura and Lee have $[f^2(\lambda)/\gamma + h(\lambda)/g(\lambda)]$. The expansion for the normal component of velocity v yields only the leading term of the matched asymptotic expansions. This anomaly is explained by observing that Sychev's results were derived allowing an error of order τ^2. Since v is already of order τ in the similarity solution, he wrote

$$u^2 \approx u^2 + v^2 = U_\infty{}^2 - [\gamma/(\gamma-1)]/(p/\rho)$$

dropping the v^2 in the result shown in Table 2. If instead we choose to retain this term of order τ^2, the resulting composite expansion for u will agree identically with that of Yakura and Lee.

From our expansion of Sychev's results, we see that his p, ρ, y, and u (with the v^2 term retained) agree identically with the matched asymptotic expansions to order τ^2. The error in his v, which is the unmodified equivalence result, is of order τ^2. Our expansions underscore the rather striking numerical agreement with Sychev which is shown in the paper of Yakura.

† Mirels pointed out the agreement for the body position $y_b = y_{(\psi=0)}$ for the case $n = \frac{1}{2}$ in his addendum.

References

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Validity of Integral Methods in MHD Boundary-Layer Analyses

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SEVERAL solutions of the MHD channel flow entrance problem[1–3] and the MHD flat-plate boundary layer[4–6] have been published recently which use the approximate momentum integral method. The attraction of the technique is the simplicity with which solutions to the nonlinear boundary-layer equations can be obtained. In view of the approximations involved in this approach, we feel that the results obtained require a more careful evaluation than they have often heretofore received before they can be accepted as being close to the exact solution to the problem. In this note we will review what we consider to be the minimum conditions under which such approximate solutions are acceptable and will ascertain the degree to which the solutions listed previously are adequate.

The reader is referred to Schlichting[7] for an excellent description of the method and for the criteria he uses to show that these approximate solutions for certain particular problems are acceptable. We can summarize these criteria as follows: 1) the approximate solution must agree well with experimental data and/or the exact, but more complicated analytical or numerical solution to the problem; 2) where no exact solution to the general problem exists, the approximate solution must agree with exact solutions for any limiting cases that may exist; and 3) the approximate solutions using several different, but physically reasonable velocity distributions (and temperature distributions for the thermal boundary layer) should show reasonable agreement with one another.

An additional requirement must be considered for MHD flow problems, since it is the differences between the behavior of the MHD flow and its ordinary hydrodynamic equivalent flow that are of the greatest interest. Thus, errors introduced by the approximations in the method should be small as compared with these differences in flow properties. Furthermore, care must be taken to base comparisons on the physically significant flow parameters; for example, for a channel flow these would be the displacement thickness δ^*, the pressure gradient dp/dx, and the friction factor f.

Laminar Momentum Boundary Layer

We now apply these criteria to the integral solution of the laminar MHD momentum boundary layer on an insulating wall.‡ For criterion 1 no experimental data are available, but finite difference solutions for MHD channel entrance flows have been obtained by Dix[8] for values of M^2/Re^2 of 10^{-2} and 10^{-3} (M is the Hartmann number), and by Shohet et al.[9] for $M = 4$. From the results of Ref. 9, values of δ^* and dp/dx can be obtained; Dix gives the wall shear stress. For criterion 2, two obvious asymptotic solutions exist. Close to the channel entrance the approximate results should be asymptotic to the Blasius solution or the ordinary hydrodynamic entrance flow solution since magnetic effects are small. At entrance lengths greater than the interaction length $\rho U/(\sigma B^2)$ the flow will become fully-developed Hartmann flow and exact analytical expressions for δ^*, dp/dx, and f can be obtained. All of these results should be used to evaluate the acceptability of the approximate solutions.

Although it is obvious from the asymptotic limits that the boundary-layer velocity profiles will not be similar along the length of the plate, two of the solutions listed assume similar velocity profiles. Maciulaitis and Loeffler[2] use a parabolic distribution that would be expected to be most appropriate for lengths that are small as compared to the interaction length. At the leading edge the friction factor for this parabolic solution differs from the exact Blasius solution by 12%. However the fully-developed solutions of Ref. 2 differ from the exact Hartmann solutions by 20%. From the data given in Ref. 2 the fully-developed value of δ/h (for $M = 10$) is 0.242, which gives $\delta^*/h = 0.081$ (compared to the Hartmann value of $1/M = 0.1$) and a fully-developed friction factor that is 0.81 times the exact Hartmann value. These differences are not apparent from this paper, because the authors compare only $u_c/U = (1 - \delta^*/h)^{-1}$ with the finite-difference solution of Shohet et al., thus obscuring the error in δ^*, and plot only the ratio of local friction factor to their own fully-developed friction factor (and not the exact Hartmann value) for comparison with the results of Dix. Moffatt[1] used the Hartmann profile for the velocity distribution in the developing boundary layer, an assumption that would be expected to give better agreement as the entrance length approaches the interaction length. His calculations of dp/dx for $M = 4$ are compared with those of Shohet et al. (difference less than 10%) and Roidt and Cess[10] (difference less than 5%) whose results are asymptotic to the Hartmann solution. At higher Hartmann numbers, the error close to the leading edge would be expected to increase.

To improve these results, the similar velocity profile assumption must be abandoned and profiles that change shape with x must be considered. Heywood[11] and Dhanak[3] have used the Pohlhausen profile (fourth-degree polynomial) in an attempt to follow the changing velocity distribution. The shape factor Λ for the MHD boundary layer is given by

$$\Lambda = \delta^2[\rho du_c/dx + \sigma B^2]/\mu \qquad (1)$$

and Heywood has shown that, when $du_c/dx = 0$, Λ increases from zero at the leading edge to a value of 12 at $x' = 0.27$. (Here x' is length along the plate as a fraction of the interaction length, $x' = x\sigma B^2/(\rho U)$; thus the boundary layer should approach the Hartmann solution for values of x' greater than unity.) At this value of Λ the solution breaks down, since for $\Lambda > 12$ the velocity profile equation yields velocities inside the boundary layer greater than the freestream value. Dhanak's solutions are discontinued at the same point and, thus, never reach the Hartmann values. However, he continues his curves as a dashed line, but does not explain why his results are discontinued nor why an asymptotic solution is never reached. This Pohlhausen profile cannot adequately follow the change in velocity distribution required by the boundary-layer forces.

A more promising profile that has been suggested by the authors assumes that the local velocity distribution is the

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‡ The flow, field configuration and notation of Ref. 1 will be adopted in the present note.